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Volatility Formulas

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1 Swaption Prices

The market practice is to quote Black's [1] volatilities instead of swaption prices, or recently due to a low interest rates, volatilities of other alternative models. For calibration of the short-rate model we need market swaption prices. The following document describes common options. The validity of the formulas used should always be confirmed with a data provider.

The computation of swaption prices requires knowledge of yield curve, zero-bond prices $P(0,T)$ in particular. The zero-bond prices are obtained from the initial yield curve model that is later used also as an input for the short-rate model.

A swaption is an option that gives buyer right to open a swap position at maturity[2]. There are two types of swaptions in terms of swap position, payer and receiver swaptions. Since the type of swaption has no impact on results of calibration, we will consider payer swaptions only.

1.1 Forward Swap Rate

We will define forward swap rate. Let T_α is swaption maturity, and $T_{\alpha+1}$, $T_{\alpha+2}$, ... T_β is increasing sequence of times, when payments of underlying swap are settled. Then

$$S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)}$$

[1] The Black's model assumes that interest rates are log-normally distributed. Therefore, the interest rates can't be negative. In recent years, the EUR interest rates were negative for shorter maturities. Therefore, the Black's model is no longer compatible with all available market data. Consequently, models based on normal distribution or shifted log-normal distribution became more common. However, for longer maturities the Black's model is still reliable.

[2] The time of option expiration will be refereed as maturity, while the lifetime of the underlying swap will be refereed as tenor.

is forward swap rate, where $\tau_i = T_i - T_{i-1}$. A swaption with a strike equal to the forward swap rate is referred to as At-The-Money (ATM hereafter).

1.2 Black's Payer Swaptions

Let $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, ..., T_\beta\}$ is a set of maturity and payment times and τ_i is time interval between individual payment times; then the payer swap price with maturity T_α, tenor $T_\beta - T_\alpha$, and strike K is

$$\mathbf{PS}^{Black}(0, \mathcal{T}, \tau, K, \sigma_{\alpha,\beta}) =$$
$$\Big[S_{\alpha,\beta}\Phi(d_1(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha})) - K\Phi(d_2(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}))\Big] \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i), \tag{1}$$

where

$$d_1(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}}{K}) + \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}}$$
$$d_2(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}}{K}) - \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}},$$

Φ is standard normal cumulative distribution function, and $\sigma_{\alpha,\beta}$ is volatility of the Black's model.

1.3 Shifted Black's Payer Swaptions

Since the Black's volatilities may not be available for shorter maturities due to the negative interest rates, the shifted Black's model may be used as an alternative. The model is exactly the same as the standard Black's model, the only difference is a shift of the log-normal distribution. Therefore; interest rates may be negative up to some threshold $-shift$. The $shift$ parameter is quoted together with Black's shifted volatilities $\sigma_{\alpha,\beta}^{shift}$, and the payer swap price is given as

$$\mathbf{PS}^{Black-shifted}(0,\mathcal{T},\tau,K,\sigma_{\alpha,\beta}^{shift},shift) = \Big[(S_{\alpha,\beta}+shift)\,\Phi(d_1(K,S_{\alpha,\beta},\sigma_{\alpha,\beta}^{shift},\sqrt{T_\alpha})) - (K+shift)\,\Phi(d_2(K,S_{\alpha,\beta},\sigma_{\alpha,\beta}^{shift},\sqrt{T_\alpha}))\Big] * \sum_{i=\alpha+1}^{\beta}\tau_i P(0,T_i), \tag{2}$$

where

$$d_1(K,S_{\alpha,\beta},\sigma_{\alpha,\beta},\sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}+shift}{K+shift}) + \frac{1}{2}\sigma_{\alpha,\beta}^2 T_\alpha}{\sigma_{\alpha,\beta},\sqrt{T_\alpha}},$$
$$d_2(K,S_{\alpha,\beta},\sigma_{\alpha,\beta},\sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}+shift}{K+shift}) - \frac{1}{2}\sigma_{\alpha,\beta}^2 T_\alpha}{\sigma_{\alpha,\beta},\sqrt{T_\alpha}}.$$

1.4 Bachelier Payer Swaptions

Another alternative to the Black's model is Bachelier model with normally distributed interest rates, and hence appropriate for economy with negative interest rates. The payer swaption price is given by

$$\mathbf{PS}^{Bachelier}(0,\mathcal{T},\tau,K,\sigma_{\alpha,\beta}^{normal}) = \sigma\sqrt{T}\left[\Phi\left(\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}}\right)\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}} + \phi\left(\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}}\right)\right],$$

where ϕ is probability density function of standard normal distribution.