TOOLS 4F

Economic Scenarios As a Service

Technical Documentation

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1 Introduction

This document presents technical details on methodology used for generating economic scenarios provided as a service (ESAS).

2 ESAS Model Description

The ESAS is based on an additive one factor and two factor Gaussian interest rate models (hereafter HW model and Additive G2++ model, respectively). The interest rate model describes the dynamics of short-rate and allows pricing of any derivative or guarantee with payoff functions that is purely dependent on interest rates. ESAS utilizes HW model as a baseline model, and the Additive G2++ model is an alternative option.[1] A key element of the short-rate model is an initial yield curve, which is modeled using the Nelson-Siegel or Nelson-Siegel-Svensson model. The model is calibrated to swaption prices, which are the most complex liquid interest rate derivatives available, i.e., their prices may be considered the most representative for calibration purposes. For the pricing of stock price dependent derivatives and guarantees, the model is extended by a geometric Brownian motion for stock prices.

This section is divided into four subsections providing details on relevant methodology. First, an econometric yield curve model for the initial yield curve is described. Second, we discuss swaption market practice and provide methodology relevant to calculating market swaption prices. Third, the general properties and computations of interest rates are discuss. Then the Hull-White and Additive G2++ models are introduced. Finally, in the last two subsections we extend the

[1] Since the financial data for model calibration provided by common data providers are usually obtained as an output from the SABR model, the HW model is mostly sufficient, while the Additive G2++ model tends to overfit the data.

interest rate model with stocks and foreign interest rates.

2.1 Yield Curve Model

The Nelson-Siegel (NS) model can be considered a special case of the Nelson-Siegel-Svensson (NSS) model; therefore, we will describe methodology for the NSS model only.

2.1.1 Zero-Yield

We will define the NSS model by equation for zero-yield, for more details see EIOPA's technical documentation. Let $R^{NSS}(0,T)$[2] is zero yield with maturity at time T, then the model is given by

$$R^{NSS}(0,T) = \beta_0 + \beta_1 \frac{1 - exp\left\{-\frac{T}{\gamma_1}\right\}}{\frac{T}{\gamma_1}} + \beta_2 \left(\frac{1 - exp\left\{-\frac{T}{\gamma_1}\right\}}{\frac{T}{\gamma_1}} - exp\left\{-\frac{T}{\gamma_1}\right\} \right) + \beta_3 \left(\frac{1 - exp\left\{-\frac{T}{\gamma_2}\right\}}{\frac{T}{\gamma_2}} - exp\left\{-\frac{T}{\gamma_2}\right\} \right),$$

where β_0, β_1, β_2, β_3, γ_0, and γ_1 are parameters. The NSS model reduces to NS model by setting $\beta_3 = 0$.

2.1.2 Zero-Bond Price

Knowing the model zero-yields, we can easily compute zero-bond prices as follows

$$P^{NSS}(0,T) = e^{-TR^{NSS}(0,T)}.$$

[2]The first argument denotes current time. In case of the initial yield curve it is always 0.

2.1.3 Instantaneous Forward Rate

Using the definition of a short-rate (i.e., interest rate paid over infinitesimally small interval) we can obtain initial forward instantaneous rates

$$f^{NSS}(0,T) = -\frac{\partial lnP(0,T)}{\partial T} = \beta_0 + \beta_1 e^{-\frac{T}{\gamma_1}} + \beta_2 T \frac{e^{-\frac{T}{\gamma_1}}}{\gamma_1} + \beta_3 T \frac{e^{-\frac{T}{\gamma_2}}}{\gamma_2}.$$

For the HW and Additive G2++ models we will also need a derivative with respect to T given by

$$\frac{\partial f^{NSS}(0,T)}{\partial T} = -\frac{\beta_1}{\gamma_1} e^{-\frac{T}{\gamma_1}} + \beta_2 \left(\frac{e^{-\frac{T}{\gamma_1}}}{\gamma_1} - T \frac{e^{-\frac{T}{\gamma_1}}}{\gamma_1^2} \right) + \beta_3 \left(\frac{e^{-\frac{T}{\gamma_2}}}{\gamma_2} - T \frac{e^{-\frac{T}{\gamma_2}}}{\gamma_2^2} \right).$$

2.1.4 Calibration

The model is calibrated to market zero-yields or to EIOPA zero-yields observed at time 0, i.e., at time of calibration of the whole ESAS model. A sum of squared errors is used as an optimization criteria. Let $R^M(0,T_i)$ are observed zero-yields with maturities T_1, T_2, ..., T_n, then the parameters are obtained as a solution to the problem

$$[\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0, \gamma_1]^T = \underset{[b_0,b_1,b_2,b_3,g_0,g_1]^T}{argmin} \sum_{i=1}^{n} \left(R^{NSS}(0,T_i) - R^M(0,T) \right)^2.$$

2.2 Swaption Prices

The market practice is to quote Black's [3] volatilities instead of swaption prices, or recently due to a low interest rates, volatilities of other alternative models. For calibration of the short-rate model we need market swaption prices.

[3] The Black's model assumes that interest rates are log-normally distributed. Therefore, the interest rates can't be negative. In recent years, the EUR interest rates were negative for shorter maturities. Therefore, the Black's model is no longer compatible with all available market data. Consequently, models based on normal distribution or shifted log-normal distribution became more common. However, for longer maturities the Black's model is still reliable.

The computation of swaption prices requires knowledge of yield curve, zero-bonds in particular. In ESAS, the zero-bond prices are obtained from the initial yield curve (NSS) model that is later used also as an input for the short-rate model.

A swaption is an option that gives buyer right to open a swap position at maturity[4]. There are two types of swaptions in terms of swap position, payer and receiver swaptions. Since the type of swaption has no impact on results of calibration, we will consider payer swaptions only.

2.2.1 Forward Swap Rate

First we will define forward swap rate. Let T_α is swaption maturity, and $T_{\alpha+1}$, $T_{\alpha+2}$, ... T_β is increasing sequence of times, when payments of underlying swap are settled. Then

$$S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i)}$$

is forward swap rate, where $\tau_i = T_i - T_{i-1}$. A swaption with a strike equal to forward swap rate is refereed as At-The-Money (ATM hereafter).

2.2.2 Black's Payer Swaptions

Let $\mathcal{T} = \{T_\alpha, T_{\alpha+1}, ..., T_\beta\}$ is a set of maturity and payment times and τ_i is time interval between individual payment times; then the payer swap price with maturity T_α, tenor $T_\beta - T_\alpha$, and strike K is

$$\mathbf{PS}^{Black}(0, \mathcal{T}, \tau, K, \sigma_{\alpha,\beta}) = \left[S_{\alpha,\beta}\Phi(d_1(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha})) - K\Phi(d_2(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}))\right] \sum_{i=\alpha+1}^{\beta} \tau_i P(0,T_i), \tag{1}$$

[4]The time of option expiration will be refereed as maturity, while the lifetime of the underlying swap will be refereed as tenor.

where

$$d_1(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}}{K}) + \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}}$$
$$d_2(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}}{K}) - \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}},$$

Φ is standard normal cumulative distribution function, and $\sigma_{\alpha,\beta}$ is volatility of the Black's model.

2.2.3 Shifted Black's Payer Swaptions

Since the Black's volatilities may not be available for shorter maturities due to the negative interest rates, the shifted Black's model may be used as an alternative. The model is exactly the same as the standard Black's model, the only difference is a shift of the log-normal distribution. Therefore; interest rates may be negative up to some threshold $-shift$. The $shift$ parameter is quoted together with Black's shifted volatilities $\sigma^{shift}_{\alpha,\beta}$, and the payer swap price is given as

$$\mathbf{PS}^{Black-shifted}(0, \mathcal{T}, \tau, K, \sigma^{shift}_{\alpha,\beta}, shift) =$$
$$\Big[(S_{\alpha,\beta} + shift)\,\Phi(d_1(K, S_{\alpha,\beta}, \sigma^{shift}_{\alpha,\beta}, \sqrt{T_\alpha})) - (K + shift)\,\Phi(d_2(K, S_{\alpha,\beta}, \sigma^{shift}_{\alpha,\beta}, \sqrt{T_\alpha}))\Big]$$
$$* \sum_{i=\alpha+1}^{\beta} \tau_i P(0, T_i), \tag{2}$$

where

$$d_1(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}+shift}{K+shift}) + \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}},$$
$$d_2(K, S_{\alpha,\beta}, \sigma_{\alpha,\beta}, \sqrt{T_\alpha}) = \frac{ln(\frac{S_{\alpha,\beta}+shift}{K+shift}) - \frac{1}{2}\sigma^2_{\alpha,\beta}T_\alpha}{\sigma_{\alpha,\beta}, \sqrt{T_\alpha}}.$$

2.2.4 Bachelier Payer Swaptions

Another alternative to the Black's model is Bachelier model with normally distributed interest rates, and hence appropriate for economy with negative interest rates. The payer swaption price is given by

$$\mathbf{PS}^{Bachelier}(0,\mathcal{T},\tau,K,\sigma_{\alpha,\beta}^{normal}) = \sigma\sqrt{T}\left[\Phi\left(\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}}\right)\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}} + \phi\left(\frac{S_{\alpha,\beta}-K}{\sigma\sqrt{T}}\right)\right],$$

where ϕ is probability density function of standard normal distribution.

2.3 Short Rate Model

Short rate model describes stochastic dynamics of the instantaneous interest rate (hereafter short-rate), i.e., interest rate paid over infinitesimally small interval. All interest rates in the economy then can be expressed as a function of short rate and its stochastic properties via no-arbitrage pricing theory. Let $r(t)$ is short-rate, then any asset at time t with payoff at time T can be priced by no-arbitrage pricing formula as

$$\mathbb{E}\left[e^{-\int_t^T r(s)ds} Payoff_T\right].$$

Before defining particular models we will define relations between short-rate, zero-bond and interest rates.

2.3.1 Zero-Bond

A zero-coupon bond with nominal value 1 at time t with maturity at time T can be priced via the no-arbitrage pricing formula as

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(s)ds} 1\right]$$

2.3.2 Zero-Yield

The zero-yield at time t with maturity at time T is given by a standard relation

$$R(t,T) = -\frac{lnP(t,T)}{T}.$$

2.3.3 Forward Rate

A model forward interest rate at time t for period from T to S is given by

$$F(t,T,S) = \frac{R(t,S)(S-t) - R(t,T)(T-t)}{S-T}.$$

2.3.4 Money-Market Account

A money-market account is a hypothetical account with one unit of currency invested at time 0 accumulating short-rate. Assume that trajectory of $r(t)$ up to time t is given, then the money-market account value at time t is

$$MM(t) = e^{\int_0^t r(u)du}.$$

The (simulated) trajectory $r(u)$ is observed only at particular times, not continuously. Therefore, the integral is calculated numerically using rectangular integration and considering left-sided values of $r(t)$. In particular, let $r(t)$ is observed in equidistant intervals with a length Δu, then the money-market account value is computed as

$$MM(t) \approx exp\left(\sum_{u=1}^{t} r(u-1)\Delta u\right).$$

2.3.5 Discount Factors

A discount-factor for a given trajectory $r(T)$ known up to time T discounting values from T to t is given as

$$DF(t,T) = e^{-\int_t^T r(u)du} = \frac{MM(t)}{MM(T)}.$$

2.4 Hull-White Model

The HW model is a short rate model, where instantaneous short rate dynamics is given by

$$\begin{aligned} dr(t) &= [\theta(t) - \alpha r(t)]dt + \sigma dW(t), \\ \theta(t) &= \frac{\partial f^M(0,t)}{\partial t} + \alpha f^M(0,t) + \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}), \end{aligned} \tag{3}$$

where W is a wiener process, f^{NSS} is given by NSS model, and α and σ are parameters. Derivations of all formulas in this section may be found in Brigo and Mercurio (2007).

2.4.1 Zero-Bond Price

Assume that trajectory of $r(t)$ is known up to time t. The price of zero-coupon bond in HW model at time t with maturity at time T is given by a formula

$$P^{HW}(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$\begin{aligned} B(t,T) &= \frac{1}{\alpha}[1 - e^{-\alpha(T-t)}], \\ A(t,T) &= \frac{P^{NSS}(0,T)}{P^{NSS}(0,t)} exp\{B(t,T)f^{NSS}(0,t) - \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha t})B(t,T)^2\}. \end{aligned}$$

2.4.2 Payer Swaption Price

Following the notation established before, let T_0 is maturity of swaption, and $T_1, T_2, ..., T_n$, $T_0 < T_1 < ... < T_n = T_\beta$ are swap payment times. Let $\mathcal{T}$ is set of maturity and payment times. The swap pays fixed strike rate K. We set $c_i := K\tau_i$ for $i = 1, 2, ..., (n-1)$ and $c_n := 1 + K\tau_i$, where τ_i is time between payments at times T_i and T_{i-1}. The payer swaption price at time 0 is then given by

$$PS^{HW}(0, \mathcal{T}, K) = \sum_{i=1}^{n} c_i ZBP(0, T_0, T_i, X_i), \tag{4}$$

where $X_i := A(T_0, T_i)exp\{-B(T_0, T_i)r^*\}$ with r^* being solution to the following equation

$$\sum_{i=1}^{n} c_i A(T_0, T_i) e^{-B(T_0,T_i)r^*} = 1,$$

and ZBP being put option price for a zero-bond at time t, with maturity at T and bond maturity at S given by formula

$$ZBP(t, T, S, X) = XP(t, T)\Phi(-h + \sigma_p) - P(t, S)\Phi(-h),$$

where

$$\sigma_p = \sigma \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, S), \tag{5}$$

$$h = \frac{1}{\sigma_p} ln \frac{P(t, S)}{P(t, T)X} + \frac{\sigma_p}{2}. \tag{6}$$

2.4.3 Model Calibration

The model is calibrated to swaption prices. The market swaption prices are obtained via Black's model (eq. (1)), shifted Black's model (eq. (2)) or Bachelier model (eq. (2.2.4)). The HW model parameters are then calibrated by solving optimization problem

$$\begin{pmatrix} \alpha \\ \sigma \end{pmatrix} = \underset{[a,s]^T}{argmin} \sum_{i=1}^{n} \left(PS_i^{Black} - PS^{HW} \right)^2.$$

2.4.4 Scenarios Generation

Once the model is calibrated, the next step is simulating the short-rate scenarios and computing required interest rates and discount factors.

Discretization of the HW short-rate dynamics (3) yields

$$r(t) = r(t-1) + [\theta(t) - \alpha r(t)] \Delta t + \sigma \sqrt{\Delta t} \epsilon^{N(0,1)}(t),$$

$$r(0) = f^{NSS}(0, 0),$$

where $\epsilon(t)$ are independent random numbers sampled from a standard normal distribution.

Zero-Bond prices, zero-yields and forward rates are functions of $r(t)$ and parameters of the model, and may be easily computed for known scenarios of short-rate using the formulas stated in the previous section.

2.5 Additive G2++ Model

The Additive G2++ model assumes that the instantaneous short rate follows

$$\begin{aligned} r(t) &= \phi(t) + x(t) + y(t), \\ dx(t) &= -ax(t)dt + \sigma dW_1(t), \\ dy(t) &= -by(t)dt + \eta dW_2(t), \\ \phi(t) &= f^{NSS}(0,t) + \frac{\sigma^2}{2a^2}\left(1-e^{-at}\right)^2 + \frac{\eta^2}{2b^2}\left(1-e^{-bt}\right)^2 + \rho\frac{\sigma\eta}{ab}\left(1-e^{-at}\right)\left(1-e^{-bt}\right), \end{aligned} \tag{7}$$

where W_1 and W_2 are Wiener processes with instantaneous correlation ρ; f^{NSS} is given by NSS model; and a, b, σ, and η are parameters. Derivations of all formulas in this section may be found in Brigo and Mercurio (2007).

2.5.1 Zero-Bond Price

Assume that trajectory of $x(t)$ and $y(t)$ is known up to time t. The price of zero-coupon bond in Additive G2++ model at time t with maturity at time T is given by formula

$$P(t,T) = exp\left\{-\int_t^T \phi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\right\},$$

where[5]

$$\begin{aligned}V(t,T) =& \frac{\sigma^2}{a^2}\left[T-t+\frac{2}{a}e^{-a(T-t)}-\frac{1}{2a}e^{-2a(T-t)}-\frac{3}{2a}\right]+\\ &\frac{\eta^2}{b^2}\left[T-t+\frac{2}{b}e^{-b(T-t)}-\frac{1}{2b}e^{-2b(T-t)}-\frac{3}{2b}\right]+\\ &2\rho\frac{\sigma\eta}{ab}\left[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b)(T-t)}-1}{a+b}\right],\\ \int_t^T\phi(u)du =& \int_t^T f^{NSS}(0,u)du+\int_t^T\frac{\sigma^2}{2a^2}\left(1-e^{-au}\right)^2du+\int_t^T\frac{\eta^2}{2b^2}\left(1-e^{-bu}\right)^2du+\\ &\int_t^T\rho\frac{\sigma\eta}{ab}\left(1-e^{-au}\right)(1-e^{-bu})du\end{aligned}$$

and

$$\begin{aligned}\int_t^T f^{NSS}(0,u)du = \beta_0(T-t)+\beta_1\gamma_1\left[e^{-\frac{t}{\gamma_1}}-e^{-\frac{T}{\gamma_1}}\right]+\beta_2\left[e^{-\frac{t}{\gamma_1}}(\gamma_1+t)-e^{-\frac{T}{\gamma_1}}(\gamma_1+T)\right]+\\ \beta_3\left[e^{-\frac{t}{\gamma_2}}(\gamma_2+t)-e^{-\frac{T}{\gamma_2}}(\gamma_2+T)\right],\\ \int_t^T\frac{\sigma^2}{2a^2}\left(1-e^{-au}\right)^2du = \frac{\sigma^2}{a^2}\frac{2a(T-t)+e^{-2at}-4e^{-at}-e^{-2aT}+4e^{-aT}}{2a},\\ \int_t^T\rho\frac{\sigma\eta}{ab}\left(1-e^{-au}\right)(1-e^{-bu})du = \rho\frac{\sigma\eta}{ab}\left[\frac{e^{-t(a+b)}-e^{-T(a+b)}}{a+b}-\frac{e^{-at}-e^{-aT}}{a}-\frac{e^{-bt}-e^{-bT}}{b}+T-t\right].\end{aligned}$$

2.5.2 Payer Swaption Price

Following the notation established before, let T_0 is maturity of swaption, and $T_1, T_2, ..., T_n$, $T_0 < T_1 < ... < T_n = T_\beta$ are swap payment times. Let $\mathcal{T}$ is set of maturity and payment times. The swap pays fixed strike rate K. We set $c_i = K\tau_i$ for $i = 1, 2, ..., (n-1)$ and $c_n = 1 + K\tau_i$, where τ_i is time between payments at times T_i and T_{i-1}. The payer swaption price at time 0 is then given by

$$PS^{G2++}(0,\mathcal{T},K) = P(0,T)\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2}}{\sigma_x\sqrt{2\pi}}\left[\Phi(-h_1(x))-\sum_{i=1}^{n}\lambda_i(x)e^{\kappa_i(x)}\Phi(-h_2(x))\right]dx, \tag{8}$$

[5]Note, the terms in $\frac{1}{2}V(t,T)$ will partially cancel-out with terms from the first integral in the zero-bond formula. However, we keep track of all stated objects for validation purposes.

where

$$h_1(x) := \frac{\bar{y} - \mu_y}{\sigma_y\sqrt{1-\rho_{xy}^2}} - \frac{\rho_{xy}(x-\mu_x)}{\sigma_x\sqrt{1-\rho_{xy}^2}},$$
$$h_2(x) := h_1(x) + B(b,T_0,T_i)\sigma_y\sqrt{1-\rho_{xy}^2},$$
$$\lambda_i(x) := c_i A(T_0,T_i)e^{-B(a,T_0,T_i)x},$$
$$\kappa_i(x) := -B(b,T_0,T_i)\left[\mu_y - \frac{1}{2}(1-\rho_{xy}^2)\sigma_y^2 B(b,T_0,T_i) + \rho_{xy}\sigma_y\frac{x-\mu_x}{\sigma_x}\right],$$

function $\bar{y} = \bar{y}(x)$ of x is a unique solution to

$$\sum_{i=1}^{n} c_i A(T_0,T_i)e^{-B(a,T_0,T_i)x - B(b,T_0,T_i)\bar{y}} = 1.$$

Furthermore

$$\mu_x := -M_x^{T_0}(0,T_0),$$
$$\mu_y := -M_y^{T_0}(0,T_0),$$
$$\sigma_x := \sigma\sqrt{\frac{1-e^{-2aT_0}}{2a}},$$
$$\sigma_y := \eta\sqrt{\frac{1-e^{-2bT_0}}{2b}},$$
$$\rho_{xy} := \frac{\rho\sigma\eta}{(a+b)\sigma_x\sigma_y}\left[1-e^{-(a+b)T_0}\right],$$
$$M_x^{T_0}(0,T_0) = \left(\frac{\sigma^2}{a^2} + \rho\frac{\sigma\eta}{ab}\right)[1-e^{-aT_0}] - \frac{\sigma^2}{2a^2}[1-e^{-2aT_0}] - \rho\frac{\sigma\eta}{b(a+b)}[1-e^{-2(a+b)T_0}].$$

Finally, the functions $A(t,T)$ and $B(z,t,T)$ are

$$A(t,T) = \frac{P^M(0,T)}{P^M(0,t)} exp\left\{\frac{1}{2}[V(t,T) - V(0,T) + V(0,t)]\right\},$$
$$B(z,t,T) = \frac{1-e^{-z(T-t)}}{z}.$$

2.5.3 Model Calibration

The model is calibrated to swaption prices. The market swaption prices are obtained via Black's model (eq. (1)), shifted Black's model (eq. (2)) or Bachelier model (eq. (2.2.4)). The Additive G2++ model parameters are then obtained by solving optimization problem

$$\begin{pmatrix} a \\ b \\ \sigma \\ \eta \\ \rho \end{pmatrix} = \underset{[b_0,b_1,b_2,b_3,g_0,g_1]^T}{argmin} \sum_{i=1}^{n} \left(PS_i^{Black} - PS^{G2++} \right)^2 .$$

2.5.4 Scenarios Generation

Once the model is calibrated, the next step is simulating the factor scenarios and computing required interest rates and discount factors.

Discretization[6] of the Ornstein-Uhlenbeck processes (7) yields

$$\begin{aligned} x(t) &= x(t-1) - ax(t-1)\Delta t + \sigma\sqrt{\Delta t}\epsilon_1^{N(0,1)}(t), \\ y(t) &= y(t-1) - by(t-1)\Delta t + \eta\sqrt{\Delta t}\left[\rho\epsilon_1^{N(0,1)}(t) + \sqrt{1-\rho^2}\epsilon_2^{N(0,1)}(t)\right], \\ x(0) &= 0, \quad y(0) = 0, \end{aligned}$$

where ϵ_1 and ϵ_2 are random numbers with independent standard normal distributions. The short-rate then can be easily obtained by its definition (7).

Zero-Bond prices, zero-yields and forward rates are functions of $x(t)$, $y(t)$ and parameters of the model, and may be easily computed for known scenarios of factors. For the details see the general subsection on short-rate models.

[6] We are using Euler discretization. Note, the more precise Milstein scheme leads to the exactly same discrete process; therefore, it is not possible to increase precision using Milstein scheme.

2.6 Pricing

The final step is using the scenarios of interest rates and discount factors to approximate the price given by the risk-neutral pricing formula. Let $Price_0$ is a price of financial derivative or guarantee at time 0 that pays $Payoff(T)$ at time T (function of known interest rates), i is index of scenario, $DF(0,T)$ is discount factor from time T to time 0, and $nsim$ is number of scenarios, then the price is approximated as

$$Price_0 = \mathbb{E}\left[DF(0,T)Payoff(T)\right] \approx \frac{1}{nsim}\sum_{i=1}^{nsim} DF_i(0,T)Payoff_i(T).$$

2.7 Equity Model

Pricing of financial derivatives or guarantees whose payoffs depend on interest rate and equities alike requires stock prices scenarios. We extend the HW model and Additive G2++ model with an equity model, where stock prices follow geometric Brownian motion. However, we will make simplifying assumption, and instead of capturing correlation between factors and stock prices, we will capture only correlation between short-rate and stock prices[7].

Let $S(t) = [S_1(t), S_2(t), ..., S_n(t)]^T$ is a price vector of n stocks, and denote

$$d\widetilde{W}(t) = \begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ \vdots \\ d\widetilde{W}_{n+1}(t) \end{pmatrix}$$

[7]We are using historical correlations. This simplifying assumption is needed for straightforward estimation of correlation coefficients between equities and interest rates. Alternatively, we could employ Kalman's filter and retrieve correlations between observed equity prices and unobserved factors of Additive G2++ model. However, short-rate dynamics frequently degenerate to a single factor models, and Kalman's filter may introduce significant error. Therefore, the simplifying assumption seems to be more suitable.

$(n+1)$ variate independent Wiener processes. Let $\psi_{i,j} \in [-1,1]$ for every i and j. Using the Cholesky decomposition we can construct $n+1$ correlated Wiener processes

$$dW(t) = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ \psi_{2,1} & \sqrt{1-\psi_{2,1}^2} & 0 & \dots & 0 \\ \psi_{3,1} & \psi_{3,2} & \sqrt{1-\psi_{3,1}^2-\psi_{3,2}^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{n,1} & \psi_{n,2} & \psi_{n,3} & \dots & \sqrt{1-\psi_{n,1}^2-\psi_{n,2}^2-\ldots-\psi_{n,n}^2} \end{pmatrix}}_{A} d\widetilde{W}(t)$$

with a correlation matrix AA^T. In the case of HW model, the first Wiener process is Wiener process from the diffusion term in the short-rate dynamics of HW model. For the Additive G2++ the first Wiener process is a Wiener process that is given by linear combination of Wiener processes as

$$dW_1(t) = \frac{\sigma dW_1^r(t) + \eta dW_2^r(t)}{(\sigma+\rho\eta)^2 + \sqrt{1-\rho^2}\eta},$$

where $dW_1^r(t)$ and $dW_2^r(t)$ are Wiener processes of $x(t)$ and $y(t)$ dynamics respectively.

Denote $dW_S(t)$ as a vector $dW(t)$ without the first element corresponding to the short-rate, then we define the equity model for n stocks allowing correlation with short-rate and between stocks as well by

$$dS(t) = (r(t) - y)\, S(t)dt + \sigma_S S(t) dW_s(t),$$

where $y = [y_1, y_2, ..., y_n]^T$ are (optional) fixed dividend rates[8] and

[8] Non-zero only if the stock pays dividends.

$$\sigma_S = \begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n \end{pmatrix}$$

is a matrix of parameters, diagonal matrix of volatilities in particular.

2.7.1 Correlation Calibration

The information about risk-neutral correlations can be obtained only from prices of complex financial derivatives that depend on both interest rates and equities at the same time. The rigorously correct approach to calibration then would be using all these derivatives to retrieve simultaneously information about whole correlation structure. Unfortunately, there are usually no liquid derivatives that would provide an information on all required correlation coefficients. Therefore, we utilize the best feasible approach available.

[HIDDEN TEXT - THIS IS PREVIEW]

2.7.2 Volatility Calibration

[HIDDEN TEXT - THIS IS PREVIEW]

2.8 FX Model

The ESAS provides FX rate scenarios, which may be used for pricing financial derivatives or guarantees whose payoffs are in foreign currencies. However, the

ESAS at this point[9] does not allow pricing with two stochastic yield curves. In other words, it is assumed that foreign interest rates are deterministic.

The FX model is methodologically identical to the Equity Model, the only difference is the continuously paid dividend, which now represents foreign instantaneous interest rate. Let $FX(t)$ is a process of one unit of domestic currency invested into a foreign currency at time $t = 0$ and governed by

$$dFX(t) = (r(t) - f_{FX}(t))FX(t)dt + \sigma_{FX}FX(t)dW_{FX}(t),$$

where σ_{FX} is parameter of the FX model, and $f_{FX}(t)$ is deterministic foreign instantaneous interest rate. The Brownian motion $dW_{FX}(t)$ may be correlated to both, short-rate and equities.

2.8.1 Calibration

[HIDDEN TEXT - THIS IS PREVIEW ONLY]

3 Economic Scenarios Validation

The scenarios are supplied with a set of validation tests that provide an evidence that the scenarios are appropriate for pricing purposes. An economic scenario generator may fail to meet the quality requirements for several reasons, which may be classified in one of the three following categories.

1. **Simulation Error.** The scenarios are used to numerically approximate the exact price given by pricing formula (see section 2.6). However, the quality of approximation depends on two parameters, a simulation step Δt and number of scenarios $nsim$. The smaller the simulation step is the finer and more precise solution of stochastic differential equations we obtain. If the step is

[9]May be extended in the future.

insufficient, it may generate bias in interest rate trajectories and therefore a bias in pricing. The number of scenarios affects stochastic convergence of approximated prices to the exact prices given by the pricing formula. Insufficient number of scenarios may again lead to a miss-pricing. This type of error is commonly checked by comparing scenario-based sample statistics with their theoretical exact counterparts.

2. **Model Error.** The real world is complex, and models used for generating economic scenarios are simplifying the real-world complexity. Therefore, too simple model may miss the important feature of the markets. For instance, market yield curve may have such shape, that can be poorly approximated by the Nielson-Siegel-Svensson model. In such case a more complex econometric model should be used[10]. The model error is typically verified by comparing model prices with real prices.

3. **Implementation Error.** The ESAS (and any other similar scenario generator) relies heavily on numerical methods and programming computational procedures. Therefore, the implementation[11] poses a risk of program failures cause by a human factor. This type of error is usually assessed by both, extensive statistical analysis of generated scenarios and comparing model prices with a market data.

The chapter is divided into three sections. First, a standard set of validation tests for the HW model is introduced, followed by the consequent second section on Additive G2++ model validation. Finally, tests related to the Equity Model extension are introduced.

[10] However, it is extremely rare that the Nielson-Siegel-Svensson model wouldn't be sufficient.
[11] Programming.

3.1 HW Model Validation

3.1.1 Martingale Test

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3.1.2 Short-rate Analysis

[HIDDEN TEXT - THIS IS PREVIEW]

3.1.3 Convergence Test

[HIDDEN TEXT - THIS IS PREVIEW]

3.2 Additive G2++ Model Validation

3.2.1 Martingale Test

[HIDDEN TEXT - THIS IS PREVIEW]

3.2.2 Short-rate Analysis

[HIDDEN TEXT - THIS IS PREVIEW]

3.2.3 Convergence Test

[HIDDEN TEXT - THIS IS PREVIEW]

3.3 Equity Model Validation

3.3.1 Martingale Test

[HIDDEN TEXT - THIS IS PREVIEW]

3.3.2 Market Consistency and Convergence

[HIDDEN TEXT - THIS IS PREVIEW]

3.4 FX Model Validation

[HIDDEN TEXT - THIS IS PREVIEW]

4 Bibliography

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2. EIOPA (2020). Technical documentation of the methodology to derive EIOPA's risk-free interest rate term structures. Available from link.